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# Evolution equations on time-dependent intervals

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## Abstract

We study initial boundary value problems for linear evolution partial differential equations (PDEs) posed on a time-dependent interval  $l_1(t) < x < l_2(t)$ ,  $0 < t < T$ , where  $l_1(t)$  and  $l_2(t)$  are given, real, differentiable functions, and  $T$  is an arbitrary constant. For such problems, we show how to characterise the unknown boundary values in terms of the given initial and boundary conditions. As illustrative examples we consider the heat equation and the linear Schrödinger equation. In the first case, the unknown Neumann boundary values are expressed in terms of the Dirichlet boundary values and of the initial value through the unique solution of a system of two linear integral equations with explicit kernels. In the second case, a similar result can be proved but only for a more restrictive class of boundary curves. Linear evolution PDE, Initial-boundary value problem, Riemann-Hilbert problem.

# 1 Introduction

We study linear evolution PDEs posed on a time-dependent interval. Specifically, we consider linear evolution PDEs on a  $t$ -dependent domain  $\Omega(t)$  of the form

$$\Omega(t) = \{(x, s) : l_1(s) < x < l_2(s), 0 < s < t\} \subset \mathbb{R}^2 \quad (1.1)$$

where  $\{l_j(t)\}_1^2$  are given, real, continuously differentiable functions, such that  $l_1(s) < l_2(s)$  for all  $s > 0$  and  $l_1(0) = 0$ ,  $l_2(0) = L \geq 0$ .

We first present some results for a general linear evolution PDE, and then concentrate on two illustrative examples, namely the heat equation and the linear Schrödinger (LS) equation:

$$(heat) \quad q_t - q_{xx} = 0, \quad (1.2)$$

$$(LS) \quad iq_t + q_{xx} = 0. \quad (1.3)$$

For a given constant  $T > 0$ , we consider the above PDEs in the domain  $\Omega(T)$  and assume that the following initial and Dirichlet boundary conditions are prescribed:

$$q(x, 0) = q_0(x) \in \mathbf{C}^1[0, L], \quad 0 < x < L, \quad (1.4)$$

$$q(l_1(t), t) = f_0(t) \in \mathbf{C}^1[0, T], \quad q(l_2(t), t) = g_0(t) \in \mathbf{C}^1[0, T], \quad 0 < t \leq T$$

We will develop our analysis in this general case, but for the particular example of the LS equation, we will need to restrict the class of boundary functions  $l_1(t)$ ,  $l_2(t)$ .

To obtain an effective representation of the solution  $q(x, t)$ , for  $t < T$ , one needs to determine the unknown Neumann boundary values  $\{q_x(l_j(t), t)\}_1^2$ , in terms of the given initial and Dirichlet boundary data  $\{q_0(x), f_0(t), g_0(t)\}$ , i.e. to characterise the so-called Dirichlet to Neumann map. The discussion and results in [Xia(2019)] imply that once this characterisation is achieved, the solution obtained is in fact the *unique* solution of the boundary value problem.

The main result of this paper is the characterisation of the Dirichlet to Neumann map for equations (1.2) and (1.3) through the unique solution of a system of two linear Volterra integral equations.

The results presented here provide a generalization of the results of [Fokas-Pelloni(2012)] (see also [DeLillo-Fokas(2007)], [Fokas-Pelloni(2007)] and [Pelloni(2008)]), where the analogous problem formulated on  $l(t) < x < \infty$  was analysed. There are two important differences between that case and the results presented here. Firstly, we must characterise two, rather than one, unknown boundary values, and hence even in the case of second order linear equations the solution is given in terms of the solution of a *system* of Volterra linear integral equations. Secondly, the kernel of the integral equations that characterise the unknown Neumann boundary value may be strongly singular. Hence to obtain a rigorous existence result this kernel must be regularised.

The main results of this paper for the two equations (1.2) and (1.3) are Theorem 1 and Theorem 2, which use the following formal representation results:

**Proposition 1 (Heat equation)** *Let  $q(x, t)$  be the solution of the heat equation (1.2) satisfying the initial and Dirichlet boundary conditions (1.4) and (1.5). Denote by  $f_1(t)$  and  $g_1(t)$  the unknown Neumann boundary values  $q_x(x, t)$  evaluated at  $x = l_1(t)$  and at  $x = l_2(t)$ :*

$$f_1(t) = q_x(l_1(t), t), \quad g_1(t) = q_x(l_2(t), t), \quad 0 < t < T. \quad (1.6)$$

*The functions  $f_1(t)$  and  $g_1(t)$  can be expressed in terms of the given initial and boundary data as the solution of the following system of coupled linear integral equations:*

$$\pi f_1(t) = N_1(t) + \int_0^t K_{11}(t, s) f_1(s) ds - \int_0^t K_{12}(t, s) g_1(s) ds, \quad 0 < t < T, \quad (1.7a)$$

$$\pi g_1(t) = N_2(t) - \int_0^t K_{22}(t, s) g_1(s) ds + \int_0^t K_{21}(t, s) f_1(s) ds, \quad 0 < t < T, \quad (1.7b)$$

*where the known functions  $N_j(t)$ ,  $j = 1, 2$ , are given by*

$$N_j(t) = \sqrt{\pi} \left\{ \frac{1}{\sqrt{t}} \int_0^L e^{-\frac{(x-l_j(t))^2}{4t}} q'_0(x) dx - \int_0^t \left[ \frac{e^{-\frac{(l_j(t)-l_1(s))^2}{4(t-s)}}}{\sqrt{t-s}} f'_0(s) - \frac{e^{-\frac{(l_j(t)-l_2(s))^2}{4(t-s)}}}{\sqrt{t-s}} g'_0(s) \right] ds \right\} \quad (1.8)$$

and the kernels  $K_{jm}(t, s)$ ,  $j, m = 1, 2$ , are given by

$$K_{jm}(t, s) = \frac{\sqrt{\pi}}{2} \frac{l_j(t) - l_m(s)}{t - s} \frac{e^{-\frac{(l_j(t) - l_m(s))^2}{4(t-s)}}}{\sqrt{t - s}}, \quad 0 < s < t < T, \quad j = 1, 2. \quad (1.9)$$

**Proposition 2 (Linear Schrödinger equation)** *Let  $q(x, t)$  be the solution of the linear Schrödinger equation (1.3) satisfying the initial and Dirichlet boundary conditions (1.4) and (1.5). Let  $f_1(t)$  and  $g_1(t)$  denote the unknown Neumann boundary values as given in expression (1.6). The unknown boundary values  $f_1(t)$  and  $g_1(t)$  can be expressed in terms of the given initial and boundary data as the solution of the following system of coupled linear integral equations:*

$$\pi f_1(t) = N_1(t) + \int_0^t K_{11}(t, s) f_1(s) ds - \lim_{\varepsilon \rightarrow 0} \int_0^t K_{12}(t, s, \varepsilon) g_1(s) ds, \quad 0 < t < T, \quad (1.10a)$$

$$\pi g_1(t) = N_2(t) + \lim_{\varepsilon \rightarrow 0} \int_0^t K_{21}(t, s, \varepsilon) f_1(s) ds - \int_0^t K_{22}(t, s) g_1(s) ds, \quad 0 < t < T, \quad (1.10b)$$

where the functions  $N_j(t)$ ,  $j = 1, 2$ , are given by

$$N_j(t) = \frac{(1-i)\sqrt{2\pi}}{2} \left\{ \frac{1}{\sqrt{t}} \int_0^L e^{\frac{i(x-l_j(t))^2}{4t}} q'_0(x) dx - \int_0^t \left[ \frac{e^{\frac{i(l_j(t)-l_1(s))^2}{4(t-s)}}}{\sqrt{t-s}} f'_0(s) - \frac{e^{\frac{i(l_j(t)-l_2(s))^2}{4(t-s)}}}{\sqrt{t-s}} g'_0(s) \right] ds \right\}. \quad (1.11)$$

The singular integral kernels  $K_{jj}(t, s)$  are given by

$$K_{jj}(t, s) = \frac{(1-i)\sqrt{2\pi}}{4} \frac{l_j(t) - l_j(s)}{t - s} \frac{e^{\frac{i(l_j(t)-l_j(s))^2}{4(t-s)}}}{\sqrt{t - s}}, \quad \varepsilon > 0, \quad 0 < s < t < T, \quad (1.12)$$

and the integral kernels  $K_{jm}(t, s, \varepsilon)$ ,  $j, m = 1, 2$ ,  $j \neq m$  are given by

$$K_{jm}(t, s, \varepsilon) = \frac{(1-i)\sqrt{2\pi}}{4} \frac{l_j(t) - l_m(s)}{t - s - i\varepsilon} \frac{e^{\frac{i(l_j(t)-l_m(s))^2}{4(t-s-i\varepsilon)}}}{\sqrt{t - s - i\varepsilon}}, \quad \varepsilon > 0, \quad 0 < s < t < T. \quad (1.13)$$

The representations above are formal, and it is not immediately clear that they actually define any function, let alone the boundary values of the given problem. In particular, in Proposition 2, we need to guarantee that the limits as  $\varepsilon \rightarrow 0$  exist, and that the Volterra integral equations are well posed and admit a unique solution.

Our main result for the heat equation is obtained by invoking classical theory for Volterra integral equations.

**Theorem 1** *For given functions  $q_0(x)$ ,  $f_0(t)$  and  $g_0(t)$  as in (1.4)-(1.5), the system of Volterra integral equations (1.7) admits a unique solution  $(f_1(t), g_1(t)) \in \mathbf{C}^1[0, T] \times \mathbf{C}^1[0, T]$ .*

For the LS equation, we need to prove that the limits as  $\varepsilon \rightarrow 0$  appearing (1.10) yield a regular kernel, and only then it will be possible to invoke classical theory.

**Theorem 2** *Assume the the boundary functions  $l_1(t)$ ,  $l_2(t)$  are twice differentiable in  $[0, T]$ , and satisfy*

$$l_1'(t) < 0, \quad l_1''(t) \geq 0; \quad l_2'(t) > 0, \quad l_2''(t) \leq 0. \quad (1.14)$$

*For given functions  $q_0(x)$ ,  $f_0(t)$  and  $g_0(t)$  as in (1.4)-(1.5), the limit system (1.10) is a system of Volterra integral equations (1.10) which admits a unique solution  $(f_1(t), g_1(t)) \in \mathbf{C}^1[0, T] \times \mathbf{C}^1[0, T]$ .*

The paper is organized as follows. In section 2, we derive formally a representation of  $q(x, t)$  for a general evolution PDE formulated in  $\Omega(T)$ , as well as the associated global relation. In section 3, we derive the formal representations of Propositions 1 and 2. Finally, in section 4 we prove Theorem 1 and Theorem 2. In the case of the linear Schrödinger equation, we also remark on the important special case of linear boundaries.

## 2 A formal integral representation for a general evolution PDE

We consider the general linear evolution PDE

$$\left( \partial_t + i \sum_{j=1}^n \alpha_j (-i \partial_x)^j \right) q(x, t) = 0, \quad (x, t) \in \Omega(T), \quad (2.1)$$

where  $\alpha_n \neq 0$  and all  $\alpha_j$ 's are constants, and the domain  $\Omega(T)$  is described by (1.1).

Let

$$\omega(\lambda) = \sum_{j=1}^n \alpha_j \lambda^j, \quad (2.2)$$

$$Q(x, t, \lambda) = - \sum_{j=1}^n \alpha_j \left( (-i \partial_x)^{j-1} + \lambda (-i \partial_x)^{j-2} + \cdots + \lambda^{j-1} \right) q(x, t). \quad (2.3)$$

The PDE (2.1) can be written in the following divergence form:

$$\left( e^{-i\lambda x + i\omega(\lambda)t} q(x, t) \right)_t = \left( e^{-i\lambda x + i\omega(\lambda)t} Q(x, t, \lambda) \right)_x. \quad (2.4)$$

Using the two-dimensional Green's theorem in the domain  $\Omega(t)$ , we obtain

$$\oint_{\partial\Omega(t)} \left( e^{-i\lambda x + i\omega(\lambda)s} q(x, s) dx + e^{-i\lambda x + i\omega(\lambda)s} Q(x, s, \lambda) ds \right) = 0, \quad 0 < t < T \quad (2.5)$$

where  $\partial\Omega(t)$  denotes the oriented boundary of the domain  $\Omega(t)$ , such that  $\Omega(t)$  lies to the left-hand side of the increasing direction. Equation (2.5) yields the relation

$$\begin{aligned} \int_0^L e^{-i\lambda x} q(x, 0) dx - e^{i\omega(\lambda)t} \int_{l_1(t)}^{l_2(t)} e^{-i\lambda x} q(x, t) dx - \int_0^t e^{-i\lambda l_1(s) + i\omega(\lambda)s} (q(l_1(s), s) l_1'(s) + Q(l_1(s), s, \lambda)) ds \\ + \int_0^t e^{-i\lambda l_2(s) + i\omega(\lambda)s} (q(l_2(s), s) l_2'(s) + Q(l_2(s), s, \lambda)) ds = 0, \quad \lambda \in \mathbb{C}, \quad 0 < t < T. \end{aligned} \quad (2.6)$$

Let

$$\hat{q}_0(\lambda) = \int_0^L e^{-i\lambda x} q(x, 0) dx, \quad (2.7a)$$

$$\hat{q}(t, \lambda) = \int_{l_1(t)}^{l_2(t)} e^{-i\lambda x} q(x, t) dx, \quad (2.7b)$$

$$\hat{Q}_1(t, \lambda) = \int_0^t e^{-i\lambda l_1(s) + i\omega(\lambda)s} (q(l_1(s), s) l_1'(s) + Q(l_1(s), s, \lambda)) ds, \quad (2.7c)$$

$$\hat{Q}_2(t, \lambda) = \int_0^t e^{-i\lambda l_2(s) + i\omega(\lambda)s} (q(l_2(s), s) l_2'(s) + Q(l_2(s), s, \lambda)) ds. \quad (2.7d)$$

Equation (2.6) can be rewritten in the form of the following global relation:

$$\hat{q}(t, \lambda) = e^{-i\omega(\lambda)t} \hat{q}_0(\lambda) - e^{-i\omega(\lambda)t} \hat{Q}_1(t, \lambda) + e^{-i\omega(\lambda)t} \hat{Q}_2(t, \lambda), \quad \lambda \in \mathbb{C}, \quad 0 < t < T. \quad (2.8)$$

Equation (2.8) can be viewed either as the formal representation of the solution, or as the starting point for determining the unknown boundary values. Indeed, the term  $\hat{q}(t, \lambda)$  in (2.8) is the Fourier transform of  $q(x, t)$  on the finite interval  $l_1(t) < x < l_2(t)$ . Inverting this Fourier transform for  $q(x, t)$ , we obtain the following formal representation of the solution:

$$q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x - i\omega(\lambda)t} [\hat{q}_0(\lambda) - \hat{Q}_1(t, \lambda) + \hat{Q}_2(t, \lambda)] d\lambda, \quad (x, t) \in \Omega(T). \quad (2.9)$$

Assuming that  $q(x, t) = 0$  for  $x < l_1(t)$  and for  $x > l_2(t)$ , equation (2.9) is also formally valid at  $x = l_1(t)$  and at  $x = l_2(t)$ . Hence evaluating the inverse Fourier transform (2.9) at these two points, we obtain

$$q(l_1(t), t) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\lambda l_1(t) - i\omega(\lambda)t} [\hat{q}_0(\lambda) - \hat{Q}_1(t, \lambda) + \hat{Q}_2(t, \lambda)] d\lambda, \quad (2.10)$$

and

$$q(l_2(t), t) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\lambda l_2(t) - i\omega(\lambda)t} [\hat{q}_0(\lambda) - \hat{Q}_1(t, \lambda) + \hat{Q}_2(t, \lambda)] d\lambda. \quad (2.11)$$

We note that in the paper [Xia(2019)] it is shown that if there exist sufficiently regular functions

$$\{q_0(x), f_0(t), g_0(t), f_1(t), g_1(t)\}$$



which satisfy the global relation, then there exists a unique regular solution  $q(x, t)$  of the PDE such that

$$\begin{aligned} q(x, 0) &= q_0(x), \quad q(l_1(t), t) = f_0(t), \quad q(l_2(t), t) = g_0(t), \\ \partial_x q(l_1(t), t) &= f_1(t), \quad \partial_x q(l_2(t), t) = g_1(t). \end{aligned} \quad (2.12)$$

Therefore if the Dirichlet to Neumann map is constructed starting from the assumption that the global relation holds, its solution does indeed provide the unique solution of the boundary value problem.

### 3 The integral equations - formal derivation

#### 3.1 The heat equation

We consider the heat equation (1.2) formulated in the time-dependent domain (1.1), with the given initial value (1.4) and Dirichlet boundary conditions (1.5). The functions  $f_1(t)$  and  $g_1(t)$ , as in (1.6), denote the unknown Neumann boundary values at  $x = l_1(t)$  and  $x = l_2(t)$  respectively.

Our aim is to determine the unknown boundary values  $f_1(t)$  and  $g_1(t)$  in terms of the given functions  $q_0(x)$ ,  $f_0(t)$  and  $g_0(t)$ , hence to characterise the Dirichlet to Neumann map. In order to determine this map, we *solve the global relation for the unknown boundary values*,  $f_1(t)$  and  $g_1(t)$ .

In the case of heat equation, in the notation of the previous section we have

$$\omega(\lambda) = -i\lambda^2, \quad Q(x, t) = q_x(x, t) + i\lambda q(x, t). \quad (3.1)$$

Then the global relation (2.8) becomes

$$\begin{aligned} \hat{q}_0(\lambda) - e^{\lambda^2 t} \hat{q}(t, \lambda) - \int_0^t e^{-i\lambda l_1(s) + \lambda^2 s} \left[ \left( i\lambda + l_1'(s) \right) q(l_1(s), s) + q_x(l_1(s), s) \right] ds \\ + \int_0^t e^{-i\lambda l_2(s) + \lambda^2 s} \left[ \left( i\lambda + l_2'(s) \right) q(l_2(s), s) + q_x(l_2(s), s) \right] ds = 0, \quad \lambda \in \mathbb{C}, \quad 0 < t < (B, 2) \end{aligned}$$

where  $\hat{q}_0(\lambda)$  and  $\hat{q}(t, \lambda)$  are defined by (2.7a) and (2.7b).

In order to derive a representation of  $q_x(x, t)$ , we follow [Fokas-Pelloni(2012)] and first multiply (3.2) by  $i\lambda$ . For the terms involving the unknown bound-

ary functions, integration by parts yields the following identity, with  $j = 1, 2$ :

$$\begin{aligned} \int_0^t e^{-i\lambda l_j(s) + \lambda^2 s} \left( -\lambda^2 + i\lambda l_j'(s) \right) q(l_j(s), s) ds &= e^{-i\lambda l_j(0)} q(l_j(0), 0) - e^{\lambda^2 t - i\lambda l_j(t)} q(l_j(t), t) \\ &+ \int_0^t e^{\lambda^2 s - i\lambda l_j(s)} \frac{dq}{ds}(l_j(s), s) ds. \end{aligned} \quad (3.3)$$

Using the formulae (3.3), equation (3.2) can be written as

$$\begin{aligned} e^{-i\lambda l_2(t)} q(l_2(t), t) - e^{-i\lambda l_1(t)} q(l_1(t), t) + i\lambda \int_{l_1(t)}^{l_2(t)} e^{-i\lambda x} q(x, t) dx &= e^{-\lambda^2 t} \left( i\lambda \hat{q}_0(\lambda) - q(0, 0) + e^{-i\lambda L} q(L, 0) \right) \\ &- e^{-\lambda^2 t} \int_0^t e^{-i\lambda l_1(s) + \lambda^2 s} \left[ \frac{dq}{ds}(l_1(s), s) + i\lambda q_x(l_1(s), s) \right] ds \\ &+ e^{-\lambda^2 t} \int_0^t e^{-i\lambda l_2(s) + \lambda^2 s} \left[ \frac{dq}{ds}(l_2(s), s) + i\lambda q_x(l_2(s), s) \right] ds. \end{aligned} \quad (3.4)$$

Using the definitions (1.4), (1.5), (1.6), and the identities

$$e^{-i\lambda l_2(t)} q(l_2(t), t) - e^{-i\lambda l_1(t)} q(l_1(t), t) + i\lambda \int_{l_1(t)}^{l_2(t)} e^{-i\lambda x} q(x, t) dx = \int_{l_1(t)}^{l_2(t)} e^{-i\lambda x} q_x(x, t) dx, \quad (3.5a)$$

$$i\lambda \hat{q}_0(\lambda) - q(0, 0) + e^{-i\lambda L} q(L, 0) = \int_0^L e^{-i\lambda x} q_0'(x) dx, \quad (3.5b)$$

equation (3.4) can be written in the form

$$\begin{aligned} \int_{l_1(t)}^{l_2(t)} e^{-i\lambda x} q_x(x, t) dx &= e^{-\lambda^2 t} \int_0^L e^{-i\lambda x} q_0'(x) dx - e^{-\lambda^2 t} \int_0^t e^{-i\lambda l_1(s) + \lambda^2 s} (f_0'(s) + i\lambda f_1(s)) ds \\ &+ e^{-\lambda^2 t} \int_0^t e^{-i\lambda l_2(s) + \lambda^2 s} (g_0'(s) + i\lambda g_1(s)) ds. \end{aligned} \quad (3.6)$$

The term on the left hand side of (3.6) is the Fourier transform of  $q_x(x, t)$  on the finite interval  $l_1(t) < x < l_2(t)$ . Inverting this Fourier transform, we obtain

$$\begin{aligned} q_x(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda x - \lambda^2 t} &\left[ \int_0^L e^{-i\lambda \xi} q_0'(\xi) d\xi - \int_0^t e^{-i\lambda l_1(s) + \lambda^2 s} (f_0'(s) + i\lambda f_1(s)) ds \right. \\ &\left. + \int_0^t e^{-i\lambda l_2(s) + \lambda^2 s} (g_0'(s) + i\lambda g_1(s)) ds \right] d\lambda, \quad (x, t) \in \Omega(\mathfrak{T}). \end{aligned} \quad (3.7)$$

Assuming that  $q_x(x, t) = 0$  for  $x < l_1(t)$  and for  $x > l_2(t)$ , equation (3.7) is also formally valid at  $x = l_1(t)$  and at  $x = l_2(t)$ . Hence, we obtain

$$\begin{aligned} f_1(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{i\lambda l_1(t) - \lambda^2 t} &\left[ \int_0^L e^{-i\lambda \xi} q_0'(\xi) d\xi - \int_0^t e^{-i\lambda l_1(s) + \lambda^2 s} (f_0'(s) + i\lambda f_1(s)) ds \right. \\ &\left. + \int_0^t e^{-i\lambda l_2(s) + \lambda^2 s} (g_0'(s) + i\lambda g_1(s)) ds \right] d\lambda, \end{aligned} \quad (3.8)$$

and

$$g_1(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{i\lambda l_2(t) - \lambda^2 t} \left[ \int_0^L e^{-i\lambda \xi} q'_0(\xi) d\xi - \int_0^t e^{-i\lambda l_1(s) + \lambda^2 s} (f'_0(s) + i\lambda f_1(s)) ds \right. \\ \left. + \int_0^t e^{-i\lambda l_2(s) + \lambda^2 s} (g'_0(s) + i\lambda g_1(s)) ds \right] d\lambda. \quad (3.9)$$

Let

$$E_j(\lambda, t, x) = e^{i\lambda(l_j(t) - x) - \lambda^2 t}, \quad j = 1, 2, \quad (3.10a)$$

$$E_{jm}(\lambda, t, s) = e^{i\lambda(l_j(t) - l_m(s)) - \lambda^2(t-s)}, \quad j, m = 1, 2. \quad (3.10b)$$

We rewrite equations (3.8) and (3.9) as

$$\pi f_1(t) = N_1(t) - i \int_{-\infty}^{\infty} \lambda \left[ \int_0^t (E_{11}(\lambda, t, s) f_1(s) - E_{12}(\lambda, t, s) g_1(s)) ds \right] d\lambda, \quad (3.11a)$$

$$\pi g_1(t) = N_2(t) - i \int_{-\infty}^{\infty} \lambda \left[ \int_0^t (E_{21}(\lambda, t, s) f_1(s) - E_{22}(\lambda, t, s) g_1(s)) ds \right] d\lambda, \quad (3.11b)$$

where

$$N_j(t) = \int_{-\infty}^{\infty} \left[ \int_0^L E_j(\lambda, t, x) q'_0(x) dx - \int_0^t (E_{j1}(\lambda, t, s) f'_0(s) - E_{j2}(\lambda, t, s) g'_0(s)) ds \right] d\lambda, \quad j = 1, 2. \quad (3.12)$$

**Claim 1** *The functions  $N_j(t)$ ,  $j = 1, 2$ , are given by (1.8).*

Interchanging the order of integration in (3.12), we find

$$N_j(t) = \int_0^L \left( \int_{-\infty}^{\infty} E_j(\lambda, t, x) d\lambda \right) q'_0(x) dx \\ - \int_0^t \left[ \left( \int_{-\infty}^{\infty} E_{j1}(\lambda, t, s) d\lambda \right) f'_0(s) - \left( \int_{-\infty}^{\infty} E_{j2}(\lambda, t, s) d\lambda \right) g'_0(s) \right] ds, \quad j = 1, 2. \quad (3.13)$$

The  $\lambda$ -integrals appearing in (3.13) can be evaluated explicitly:

$$\int_{-\infty}^{\infty} E_j(\lambda, t, x) d\lambda = \frac{\sqrt{\pi}}{\sqrt{t}} e^{-\frac{(x - l_j(t))^2}{4t}}, \quad j = 1, 2, \quad (3.14a)$$

$$\int_{-\infty}^{\infty} E_{jm}(\lambda, t, s) d\lambda = \frac{\sqrt{\pi}}{\sqrt{t-s}} e^{-\frac{(l_j(t) - l_m(s))^2}{4(t-s)}}, \quad j, m = 1, 2. \quad (3.14b)$$

Substituting the above expressions into (3.13), we immediately obtain the formulae (1.8).

**Claim 2** For a given function  $h(s) \in \mathbf{C}[0, T]$ , the following identities hold:

$$-i \int_{-\infty}^{\infty} \lambda \int_0^t E_{jj}(\lambda, t, s) h(s) ds d\lambda = \int_0^t K_{jj}(t, s) h(s) ds, \quad (3.15)$$

where  $K_{jj}(t, s)$ ,  $j = 1, 2$ , as given by (1.9), is a weakly singular kernel;

$$-i \int_{-\infty}^{\infty} \lambda \int_0^t E_{jm}(\lambda, t, s) h(s) ds d\lambda = \int_0^t K_{jm}(t, s) h(s) ds, \quad (3.16)$$

where  $K_{jm}(t, s)$  are the non-singular integral kernels given by (1.9) when  $j \neq m$ .

To show both claims, we formally interchange the order of double integration in the left hand side of (3.15) and (3.16). We will justify the validity of this procedure by showing that all integrands are integrable.

By definition,

$$-i \int_{-\infty}^{\infty} \lambda \int_0^t E_{jm}(\lambda, t, s) h(s) ds d\lambda = \int_{-\infty}^{\infty} \lambda \int_0^t -i e^{-\lambda^2(t-s) + i\lambda(l_j(t) - l_m(s))} h(s) ds d\lambda.$$

We now interchange the order of integration, and use integration by parts for the inner integral to obtain

$$\begin{aligned} -i \int_{-\infty}^{\infty} \lambda e^{-\lambda^2(t-s) + i\lambda(l_j(t) - l_m(s))} d\lambda &= \frac{i}{2(t-s)} \int_{-\infty}^{\infty} \left( \frac{\partial}{\partial \lambda} e^{-\lambda^2(t-s) + i\lambda(l_j(t) - l_m(s))} \right) d\lambda \\ &+ \frac{l_j(t) - l_m(s)}{2(t-s)} \int_{-\infty}^{\infty} e^{-\lambda^2(t-s) + i\lambda(l_j(t) - l_m(s))} d\lambda, \quad j, m = 1, 2. \end{aligned} \quad (3.17)$$

The first integral of the right hand side of the above equation vanishes, whereas the second integral can be computed explicitly as (see (3.14b))

$$\frac{l_j(t) - l_m(s)}{2(t-s)} \int_{-\infty}^{\infty} e^{-\lambda^2(t-s) + i\lambda(l_j(t) - l_m(s))} d\lambda = \frac{l_j(t) - l_m(s)}{2(t-s)} \frac{\sqrt{\pi}}{\sqrt{t-s}} e^{-\frac{(l_j(t) - l_m(s))^2}{4(t-s)}}, \quad j, m = 1, 2.$$

which is the kernel  $K_{jm}(t, s)$  as defined by equation (1.9).

Note that

**If  $j = m$ :**

In this case, the singularity at  $s = t$  due to the term  $\frac{1}{t-s}$  is removable, as

$$\lim_{s \rightarrow t} \frac{l_j(t) - l_j(s)}{2(t-s)} = \frac{1}{2} l'_j(t).$$

Hence the kernel  $K_{jj}$  has the weak, integrable singularity  $\frac{1}{\sqrt{t-s}}$ .

**If  $j \neq m$ :**

In this case, the singularity  $\frac{1}{(t-s)^{3/2}}$  is removable as it is cancelled by the zero of the exponential term  $e^{-\frac{(l_j(t)-l_m(s))^2}{4(t-s)}}$ . Therefore the kernel  $K_{jm}$ ,  $j \neq m$ , is regular at  $s = t$ .

Collecting the results of these claims, equations (3.12) yield the system (1.7).

**Remark 1** *Our requirement in Claim 2 that  $h \in C[0, T]$  is sufficient to justify the interchange in the integration order, and it is all we need for our purposes, as this is the class of the unknown boundary values we seek to determine.*

### 3.2 The linear Schrödinger equation

We consider the linear Schrödinger equation (1.3). As for the case of the heat equation considered in the previous section, to determine the unknown Neumann boundary values  $f_1(t)$  and  $g_1(t)$  given in (1.6) in terms of the given functions  $q_0(x)$ ,  $f_0(t)$  and  $g_0(t)$  we solve the global relation for the unknown boundary values  $f_1(t)$  and  $g_1(t)$ .

In the case of linear Schrödinger equation, we have

$$\omega(\lambda) = \lambda^2, \quad Q(x, t) = iq_x(x, t) - \lambda q(x, t). \quad (3.18)$$

Hence the global relation (2.8) becomes

$$\begin{aligned} & \hat{q}_0(\lambda) - e^{i\lambda^2 t} \hat{q}(t, \lambda) - \int_0^t e^{-i\lambda l_1(s) + i\lambda^2 s} \left[ \left( l_1'(s) - \lambda \right) q(l_1(s), s) + iq_x(l_1(s), s) \right] ds \\ & + \int_0^t e^{-i\lambda l_2(s) + i\lambda^2 s} \left[ \left( l_2'(s) - \lambda \right) q(l_2(s), s) + iq_x(l_2(s), s) \right] ds = 0, \quad \lambda \in \mathbb{C}, \quad 0 < t \leq T \end{aligned} \quad (3.19)$$

In analogy with the case of the heat equation, in order to obtain a representation of  $q_x(x, t)$ , we multiply (3.19) by  $i\lambda$  and then employ integration by parts for the terms involving the known functions  $q(l_1(t), t)$  and  $q(l_2(t), t)$ .

In this way, equation (3.19) yields the equation

$$\begin{aligned} e^{-i\lambda l_2(t)} q(l_2(t), t) - e^{-i\lambda l_1(t)} q(l_1(t), t) + i\lambda \int_{l_1(t)}^{l_2(t)} e^{-i\lambda x} q(x, t) dx = e^{-i\lambda^2 t} \left( i\lambda \hat{q}_0(\lambda) - q(0, 0) + e^{-i\lambda L} q(L, 0) \right) \\ - e^{-i\lambda^2 t} \int_0^t e^{-i\lambda l_1(s) + i\lambda^2 s} \left[ \frac{dq}{ds}(l_1(s), s) - \lambda q_x(l_1(s), s) \right] ds + e^{-i\lambda^2 t} \int_0^t e^{-i\lambda l_2(s) + i\lambda^2 s} \left[ \frac{dq}{ds}(l_2(s), s) - \lambda q_x(l_2(s), s) \right] ds. \end{aligned} \quad (3.20)$$

Employing the identities (3.5), equation (3.20) can be written in the form

$$\begin{aligned} \int_{l_1(t)}^{l_2(t)} e^{-i\lambda x} q_x(x, t) dx = e^{-i\lambda^2 t} \int_0^L e^{-i\lambda \xi} q'_0(\xi) d\xi - e^{-i\lambda^2 t} \int_0^t e^{-i\lambda l_1(s) + i\lambda^2 s} \left[ \frac{dq}{ds}(l_1(s), s) - \lambda q_x(l_1(s), s) \right] ds \\ + e^{-i\lambda^2 t} \int_0^t e^{-i\lambda l_2(s) + i\lambda^2 s} \left[ \frac{dq}{ds}(l_2(s), s) - \lambda q_x(l_2(s), s) \right] ds. \end{aligned} \quad (3.21)$$

The term on the left hand side of (3.21) is the Fourier transform of  $q_x(x, t)$ . Inverting this Fourier transform of  $q_x(x, t)$  and evaluating it at  $x = l_j(t)$ ,  $j = 1, 2$ , we arrive at

$$\begin{aligned} & q_x(l_j(t), t) \\ = & \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\lambda l_j(t)} \left[ e^{-i\lambda^2 t} \int_0^L e^{-i\lambda x} q'_0(x) dx - e^{-i\lambda^2 t} \int_0^t e^{-i\lambda l_1(s) + i\lambda^2 s} \left( \frac{dq}{ds}(l_1(s), s) - \lambda q_x(l_1(s), s) \right) ds \right. \\ & \left. + e^{-i\lambda^2 t} \int_0^t e^{-i\lambda l_2(s) + i\lambda^2 s} \left( \frac{dq}{ds}(l_2(s), s) - \lambda q_x(l_2(s), s) \right) ds \right] d\lambda, \quad j = 1, 2. \end{aligned} \quad (3.22)$$

Setting

$$\mathcal{E}_j(\lambda, t, x) = e^{i\lambda(l_j(t) - x) - i\lambda^2 t}, \quad j = 1, 2, \quad (3.23a)$$

$$\mathcal{E}_{jm}(\lambda, t, s) = e^{i\lambda(l_j(t) - l_m(s)) - i\lambda^2(t - s)}, \quad j, m = 1, 2, \quad (3.23b)$$

and using (1.4), (1.5) and (1.6), we write equation (3.22) in the form

$$\pi f_1(t) = N_1(t) + \int_{-\infty}^{\infty} \lambda \left[ \int_0^t (\mathcal{E}_{11}(\lambda, t, s) f_1(s) - \mathcal{E}_{12}(\lambda, t, s) g_1(s)) ds \right] d\lambda, \quad (3.24a)$$

$$\pi g_1(t) = N_2(t) + \int_{-\infty}^{\infty} \lambda \left[ \int_0^t (\mathcal{E}_{21}(\lambda, t, s) f_1(s) - \mathcal{E}_{22}(\lambda, t, s) g_1(s)) ds \right] d\lambda, \quad (3.24b)$$

where, for  $j = 1, 2$ ,

$$N_j(t) = \int_{-\infty}^{\infty} \left[ \int_0^L \mathcal{E}_j(\lambda, t, x) q'_0(x) dx - \int_0^t (\mathcal{E}_{j1}(\lambda, t, x) f'_0(s) - \mathcal{E}_{j2}(\lambda, t, x) g'_0(s)) ds \right] d\lambda \quad (3.25)$$

We want to rewrite (3.24) in the form (1.10).

**Claim 3** *The functions  $N_j(t)$ ,  $j = 1, 2$ , are given by (1.11).*

Interchanging the order of integration in (3.25), we find

$$\begin{aligned} N_j(t) &= \int_0^L \left( \int_{-\infty}^{\infty} \mathcal{E}_j(\lambda, t, x) d\lambda \right) q'_0(x) dx \\ &\quad - \int_0^t \left[ \left( \int_{-\infty}^{\infty} \mathcal{E}_{j1}(\lambda, t, s) d\lambda \right) f'_0(s) - \left( \int_{-\infty}^{\infty} \mathcal{E}_{j2}(\lambda, t, s) d\lambda \right) g'_0(s) \right] ds, \quad j = 1, 2. \end{aligned} \quad (3.26)$$

The  $\lambda$ -integrals appearing in (3.26) can be evaluated explicitly:

$$\int_{-\infty}^{\infty} \mathcal{E}_j(\lambda, t, x) d\lambda = \frac{(1-i)\sqrt{2\pi}}{2\sqrt{t}} e^{\frac{i(x-l_j(t))^2}{4t}}, \quad j = 1, 2, \quad (3.27a)$$

$$\int_{-\infty}^{\infty} \mathcal{E}_{jm}(\lambda, t, s) d\lambda = \frac{(1-i)\sqrt{2\pi}}{2\sqrt{t-s}} e^{\frac{i(l_j(t)-l_m(s))^2}{4(t-s)}}, \quad j, m = 1, 2. \quad (3.27b)$$

Substituting (3.27) into (3.26) we immediately obtain the formulae (1.11).

**Claim 4** *For a given function  $h(s) \in \mathbf{C}[0, T]$ , the following identities hold:*

$$\int_{-\infty}^{\infty} \lambda \int_0^t \mathcal{E}_{jj}(\lambda, t, s) h(s) ds d\lambda = \int_0^t K_{jj}(t, s) h(s) ds, \quad (3.28)$$

$$\int_{-\infty}^{\infty} \lambda \int_0^t \mathcal{E}_{jm}(\lambda, t, s) h(s) ds d\lambda = \lim_{\varepsilon \rightarrow 0} \int_0^t K_{jm}(t, s, \varepsilon) h(s) ds, \quad (3.29)$$

where  $K_{jj}(t, s)$ ,  $K_{jm}(s, t, \varepsilon)$   $j, m = 1, 2$ , are the integral kernels given by (1.12) and (1.13) respectively.

If we interchange the order of double integration in the left hand side of (3.28), we obtain an integrable functions of  $s$ . However, for the case  $j \neq m$  of (3.29), we obtain a function that is not integrable with respect to  $s$ . Thus,

before interchanging the order of the integration we must first regularise the relevant  $\lambda$ -integral. Therefore, we write

$$\begin{aligned} \int_{-\infty}^{\infty} \lambda \mathcal{E}_{jm}(\lambda, t, s) h(s) ds d\lambda &= \int_{-\infty}^{\infty} \lambda \int_0^t \lim_{\varepsilon \rightarrow 0^+} e^{-\varepsilon \lambda^2 - i\lambda^2(t-s) + i\lambda(l_j(t) - l_1(s))} h(s) ds d\lambda \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \lambda \int_0^t e^{-\varepsilon \lambda^2 - i\lambda^2(t-s) + i\lambda(l_j(t) - l_1(s))} h(s) ds d\lambda, \end{aligned} \quad (3.30)$$

where the last identity follows from the dominated convergence theorem, thanks to the exponential decay of the term  $e^{-\varepsilon \lambda^2}$ . Now we can interchange the order of integration, hence the expression in (3.30) is equal to

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_0^t K_{jm}(t, s, \varepsilon) h(s) ds, \quad K_{jm}(t, s, \varepsilon) &= \int_{-\infty}^{\infty} \lambda e^{-i\lambda^2(t-s-i\varepsilon) + i\lambda(l_j(t) - l_m(s))} d\lambda, \\ j, m &= 1, 2, j \neq m \end{aligned} \quad (3.31)$$

The  $\lambda$ -integral (3.31) can be evaluated explicitly. Indeed, the  $\lambda$ -derivative of the exponent of the exponential appearing in (3.31) is given by the expression:

$$[-2i\lambda(t-s-i\varepsilon) + i(l_j(t) - l_m(s))] e^{-i\lambda^2(t-s-i\varepsilon) + i\lambda(l_j(t) - l_m(s))}.$$

Hence, we can rewrite  $K_{jm}(t, s, \varepsilon)$  in the form

$$\begin{aligned} K_{jm}(t, s, \varepsilon) &= \frac{-1}{2i(t-s-i\varepsilon)} \int_{-\infty}^{\infty} \left( \frac{\partial}{\partial \lambda} e^{-i\lambda^2(t-s-i\varepsilon) + i\lambda(l_j(t) - l_m(s))} \right) d\lambda \\ &\quad + \frac{l_j(t) - l_m(s)}{2(t-s-i\varepsilon)} \int_{-\infty}^{\infty} e^{-i\lambda^2(t-s-i\varepsilon) + i\lambda(l_j(t) - l_m(s))} d\lambda, \quad j, m = 1, 2, j \neq m \end{aligned} \quad (3.32)$$

The first integral of the right hand side of the above equation vanishes, because of the large  $\lambda$  decay of the term  $e^{-\lambda^2 \varepsilon}$ , whereas the second integral can be computed explicitly (see (3.27b)). This yields the expression

$$K_{jm}(t, s, \varepsilon) = \frac{l_j(t) - l_m(s)}{2(t-s-i\varepsilon)} \frac{(1-i)\sqrt{2\pi}}{2\sqrt{t-s-i\varepsilon}} e^{\frac{i(l_j(t) - l_m(s))^2}{4(t-s-i\varepsilon)}}, \quad j, m = 1, 2, j \neq m \quad (3.33)$$

In summary, using (3.30)-(3.32), we find

$$\int_{-\infty}^{\infty} \lambda \mathcal{E}_{jm}(\lambda, t, s) h(s) ds = \frac{(1-i)\sqrt{2\pi}}{4} \lim_{\varepsilon \rightarrow 0^+} \int_0^t \frac{l_j(t) - l_m(s)}{(t-s-i\varepsilon)^{3/2}} e^{\frac{i(l_j(t) - l_m(s))^2}{4(t-s-i\varepsilon)}} h(s) ds, \quad (3.34)$$

yielding the conclusion of Proposition 2.



## 4 Proof of the main theorems

We now sketch the step needed to prove that the representation derived in the previous section yields Volterra integral equations that admit a unique solution. The only new ingredient in this section is the analysis of the  $\varepsilon \rightarrow 0$  limit of the integrals appearing in the representation given by Proposition 2.

After proving that these limits yield a well defined system of Volterra integral equations, possibly weakly singular, the proof is analogous to the proof given in [Fokas-Pelloni(2012)] for the problem formulated on  $l(t) < x < \infty$ . We refer to these papers for details, and concentrate on showing that the integral equations derived in the previous section are of a type that can be treated using classical results.

We note that for both the case of the heat equation and the linear Schrödinger equation, the case that  $j = m$  yields a weakly singular kernel.

Indeed, in this case, the singularity at  $s = t$  due to the term  $\frac{1}{t-s}$  is removable, as

$$\lim_{s \rightarrow t} \frac{l_j(t) - l_j(s)}{2(t-s)} = \frac{1}{2} l'_j(t).$$

Hence the kernel  $K_{jj}$  has the weak, integrable singularity  $\frac{1}{\sqrt{t-s}}$ .

However, if  $j \neq m$ , this is not the case. We consider the two theorems separately.

### 4.1 Theorem 1

For the kernel given by expression (1.9), the singularity  $\frac{1}{(t-s)^{3/2}}$  is removable as it is cancelled by the zero of the exponential term  $e^{-\frac{(l_j(t)-l_m(s))^2}{4(t-s)}}$ . Therefore the kernel  $K_{jm}$ ,  $j \neq m$ , is regular at  $s = t$ .

Under our regularity assumptions on the known data, it follows that the system of Volterra integral equations (1.7) admits a unique solution. The proof is identical to the proof given in [Fokas-Pelloni(2012)] and relies on general results for Volterra integral equations with weakly singular kernels, given e.g. in [Miller-Feldstein(1971)].

## 4.2 Theorem 2

Next we consider the kernel given by expression (3.33). For  $\varepsilon > 0$ , this kernel has no singularity, hence invoking again the general results of [Fokas-Pelloni(2012), Miller-Feldstein(1971)] we can deduce that the vector Volterra integral equation

$$\begin{pmatrix} f_1^\varepsilon(t) \\ g_1^\varepsilon(t) \end{pmatrix} = \begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix} + \int_0^t \begin{pmatrix} K_{11}(t, s) & -K_{12}(t, s, \varepsilon) \\ K_{21}(t, s, \varepsilon) & -K_{22}(t, s) \end{pmatrix} \begin{pmatrix} f_1^\varepsilon(s) \\ g_1^\varepsilon(s) \end{pmatrix} ds \quad (4.1)$$

admits a unique solution  $(f_1^\varepsilon, g_1^\varepsilon) \in \mathbf{C}[0, T) \times \mathbf{C}[0, T)$  for every  $\varepsilon > 0$ .

The last step in the proof is the consideration of the  $\varepsilon \rightarrow 0^+$  limit in the expression above. It must be shown that this limit exists. To avoid technicalities and focus on the essential issue of the  $\varepsilon \rightarrow 0$  limit, we show this for the analogous scalar case - the extension to the case of the vector integral equation (4.1) is immediate.

We first consider the  $\varepsilon$ -dependent kernel  $K_{12}$ . Recall that, for a function  $h(s) : [0, t] \rightarrow \mathbb{R}$  bounded and sufficiently regular,

$$\int_0^t K_{12}(t, s, \varepsilon) h(s) ds = \frac{(1-i)\sqrt{2\pi}}{4} \int_0^t \frac{l_1(t) - l_2(s)}{(t-s-i\varepsilon)^{3/2}} e^{\frac{i(l_1(t)-l_2(s))^2}{4(t-s-i\varepsilon)}} h(s) ds. \quad (4.2)$$

We now consider the exponential appearing in the integrand

$$\mathcal{E}_{12}(t, s, \varepsilon) = e^{\frac{i(l_1(t)-l_2(s))^2}{4(t-s-i\varepsilon)}}$$

Differentiating  $\mathcal{E}_{12}$  with respect to  $s$ , and rearranging, we can write the integrand in (4.2) as

$$\frac{l_1(t) - l_2(s)}{(t-s-i\varepsilon)^{3/2}} \mathcal{E}_{12}(t, s, \varepsilon) = -4i \frac{(t-s-i\varepsilon)^{1/2}}{l_1(t) - l_2(s) - 2l_2'(s)(t-s-i\varepsilon)} \frac{\partial \mathcal{E}_{12}}{\partial s}.$$

Hence

$$\int_0^t K_{12}(t, s, \varepsilon) h(s) ds = -(i+1)\sqrt{2\pi} \int_0^t \frac{(t-s-i\varepsilon)^{1/2}}{l_1(t) - l_2(s) - 2l_2'(s)(t-s-i\varepsilon)} \frac{\partial \mathcal{E}_{12}}{\partial s} h(s) ds.$$

Integration by parts yields

$$-\frac{1}{(i+1)\sqrt{2\pi}} \int_0^t K_{12}(t, s, \varepsilon) h(s) ds =$$

$$\begin{aligned} & \frac{(-i\varepsilon)^{1/2}}{l_1(t) - l_2(t) + 2l'_2(t)i\varepsilon} h(t) \mathcal{E}_{12}(t, t, \varepsilon) - \frac{(t - i\varepsilon)^{1/2}}{l_1(t) - l_2(0) - 2l'_2(0)(t - i\varepsilon)} h(0) \mathcal{E}_{12}(t, 0, \varepsilon) \\ & - \int_0^t \mathcal{E}_{12}(t, s, \varepsilon) \left\{ \frac{-\frac{h(s)}{2(t-s-i\varepsilon)^{1/2}} + (t-s-i\varepsilon)^{1/2} h'(s)}{l_1(t) - l_2(s) - 2l'_2(s)(t-s-i\varepsilon)} - \frac{(t-s-i\varepsilon)^{1/2} [l'_2(s) - 2l''_2(s)(t-s-i\varepsilon)] h(s)}{(l_1(t) - l_2(s) - 2l'_2(s)(t-s-i\varepsilon))^2} \right\} ds. \end{aligned}$$

We now need to take the limit as  $\varepsilon \rightarrow 0$ ; in order to pass to the limit inside the integral on the right hand side using the dominated convergence theorem, we must show that the integrand is dominated by an integrable function.

Let

$$H_1(t, s) = l_1(t) - l_2(s) - 2l'_2(s)(t - s). \quad (4.3)$$

If  $H_1(t, s) \neq 0$  for all  $s \in [0, t]$ , then the integrand can be dominated by  $g(s) = \frac{c}{\sqrt{t-s}}$ , and this function is integrable in  $[0, t]$ . Therefore, under this assumption, that we will return to below, we can pass to limit under the integral and we find

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{(i+1)\sqrt{2\pi}} \int_0^t K_{12}(t, s, \varepsilon) h(s) ds &= \frac{t^{1/2}}{H_1(t, 0)} h(0) \mathcal{E}_{12}(t, 0, 0) + \int_0^t \mathcal{E}_{12}(t, s, 0) \frac{(t-s)^{1/2}}{H_1(t, s)} h'(s) ds \\ &- \int_0^t \mathcal{E}_{12}(t, s, 0) \left\{ \frac{1}{2(t-s)^{1/2} H_1(t, s)} + \frac{[l'_2(s) - 2l''_2(s)(t-s)] (t-s)^{1/2}}{H_1(t, s)^2} \right\} h(s) ds. \end{aligned}$$

Thus this limit has the form

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{(i+1)\sqrt{2\pi}} \int_0^t K_{12}(t, s, \varepsilon) h(s) ds &= M_{12}^h(t) - \int_0^t \mathcal{E}_{12}(t, s) \frac{1}{H_1(t, s)} \left[ \frac{1}{2(t-s)^{1/2}} + \frac{[l'_2(s) - 2l''_2(s)(t-s)] (t-s)^{1/2}}{H_1(t, s)} \right. \\ &\left. + \int_0^t \mathcal{E}_{12}(t, s) (t-s)^{1/2} \frac{1}{H_1(t, s)} h'(s) ds \right] \end{aligned}$$

with

$$M_{12}^h(t) = \mathcal{E}_{12}(t, 0, 0) t^{1/2} \frac{1}{H_1(t, 0)} h(0)$$

and  $H_1(t, s)$  given by (4.3). Hence for  $f_1(t) = \lim_{\varepsilon \rightarrow 0} f_1^\varepsilon(t)$ ,  $g_1(t) = \lim_{\varepsilon \rightarrow 0} g_1^\varepsilon(t)$ ,

using equation (4.1) we find the integral equation

$$\begin{aligned}
f_1(t) &= N_1(t) - (i+1)\sqrt{2\pi}M_{12}^{g_1}(t) + \int_0^t K_{11}(t,s)f_1(s)ds \\
&+ (i+1)\sqrt{2\pi} \left\{ \int_0^t \mathcal{E}_{12}(t,s) \frac{1}{H_1(t,s)} \left[ \frac{1}{2(t-s)^{1/2}} + (t-s)^{1/2} \frac{[l_2'(s) - 2l_2''(s)(t-s)](t-s)^{1/2}}{H_1(t,s)} \right] g_1(s)ds \right. \\
&\left. - \int_0^t \mathcal{E}_{12}(t,s)(t-s)^{1/2} \frac{1}{H_1(t,s)} g_1'(s)ds \right\}, \tag{4.4}
\end{aligned}$$

where  $N_1(t)$  is given by (1.11).

An analogous computation for the kernel  $K_{21}$  yields

$$\begin{aligned}
g_1(t) &= N_2(t) - (i+1)\sqrt{2\pi}M_{21}^{f_1}(t) - \int_0^t K_{22}(t,s)g_1(s)ds \\
&- (i+1)\sqrt{2\pi} \left\{ \int_0^t \mathcal{E}_{21}(t,s) \frac{1}{H_2(t,s)} \left[ \frac{1}{2(t-s)^{1/2}} + \frac{[l_1'(s) - 2l_1''(s)(t-s)](t-s)^{1/2}}{H_2(t,s)} \right] f_1(s)ds \right. \\
&\left. - \int_0^t \mathcal{E}_{21}(t,s)(t-s)^{1/2} \frac{1}{H_2(t,s)} f_1'(s)ds \right\}, \tag{4.5}
\end{aligned}$$

with  $N_2(t)$  given by (1.11),

$$\mathcal{E}_{21}(t,s,\varepsilon) = e^{\frac{i(l_2(t)-l_1(s))^2}{4(t-s-i\varepsilon)}}, \quad M_{21}^h(t) = \mathcal{E}_{21}(t,0,0)t^{1/2} \frac{1}{H_2(t,0)} h(0)$$

and

$$H_2(t,s) = l_2(t) - l_1(s) - 2l_1'(s)(t-s).$$

We claim that the two equations (4.4)-(4.5) above are a system of a generalised Volterra integral equation of the second kind with a weakly singular kernel.

We first note that these equations are not in the usual form of a Volterra integral equation for the functions  $f(t)$ ,  $g(t)$ , since the right hand side contains not only the functions but also their first derivative. A modification of the iterative proof of existence of a solution for the usual Volterra case also works in this generalised case, see [Brunner(2017)].

It remains to prove that the kernels appearing in the two integral on the right hand side of (4.4)-(4.5) are weakly singular. This is clearly the case provided  $H_1(t,s)$  and  $H_2(t,s)$  do not vanish for any  $s \in [0,t]$ .

For  $H_1(t, s)$  the condition is that

$$l_1(t) - l_2(s) - 2l'_2(s)(t - s) \neq 0, \quad \forall 0 \leq s \leq t.$$

Since  $l_1(t) < l_2(t)$ , if we assume that  $l_2(t)$  satisfies the condition (1.14) we have

$$l_1(t) - l_2(s) - 2l'_2(s)(t - s) < l_2(t) - l_2(s) - 2l'_2(s)(t - s) = (t - s)[l'_2(t) - 2l'_2(s)] < 0, \quad s \leq \sigma \leq t.$$

Similarly, for  $H_2(t, s)$  the condition is

$$l_2(t) - l_1(s) - 2l'_1(s)(t - s) \neq 0, \quad \forall 0 \leq s \leq t.$$

and if  $l_1(t)$  satisfies the condition (1.14) we have

$$l_2(t) - l_1(s) - 2l'_1(s)(t - s) > l_1(t) - l_1(s) - 2l'_1(s)(t - s) > 0.$$

Hence under the assumption (1.14), the regularity condition is satisfied for both  $K_{12}$  and  $K_{21}$ . Hence the assumption (1.14) is sufficient to ensure the sought regularity. This completes the proof of the theorem.

**Remark 2 (Linear boundaries)** *The case when boundaries are linear is often of interest in applications. In this case, we can prove the main result without the need to assume equations (1.14). This shows that (1.14) is a sufficient but not necessary condition.*

*Indeed, assume that the boundary curves are of the form*

$$l_1(s) = \alpha s, \quad l_2(s) = \beta s + L, \quad 0 < s < t, \quad 0 < 2\alpha < \beta, \quad L \geq 0. \quad (4.6)$$

*For  $H_1(t, s)$  the non-zero condition becomes*

$$(\alpha - 2\beta)t + \beta s - L \neq 0 \iff s \neq \frac{2\beta - \alpha}{\beta}t + \frac{L}{\beta}.$$

*Since  $s \leq t$ , this always holds if  $\beta > \alpha > 0$ . For  $H_2(t, s)$ , under the assumption (4.6), the non-zero condition is*

$$(\beta - 2\alpha)t + \alpha s + L \neq 0$$

*which is always true if all terms are positive, i.e. if  $\beta > 2\alpha > 0$ .*

*Therefore, Theorem (2) is valid for linear boundaries of the form (4.6).*

## 5 Conclusions

We have shown how to give a solution representation for boundary value problems for linear evolution equation in one spatial variable, posed between two time-dependent boundaries, issuing from the common point set at the origin of the  $(x, t)$  plane.

For the specific example of the heat equation, the solution is obtained as the unique solution of a system of Volterra integral equations (1.7), valid for any choice of differentiable boundary curves not intersecting for positive times.

For the case of the linear Schrödinger equation, the solution is again given as the unique solution of the system of generalised Volterra integral equations (4.4)-(4.5), but only for a more restricted class of boundaries, satisfying condition (1.14) or (4.6).

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